Alexandros Droseltis
18/12/2011

Ito formula as special and general case

Ito Formula is related in 2 ways with standard analysis. In one way it is a more general chain rule for differentiation. In on other way it is a special case of a Taylor expansion. In this post I will demonstrate informally both approaches. But before doing this, let us give the simplest form of the Ito formula.

Let W_t be a Brownian Motion and define the stochastic process S over

$$dS_t = \mu dt + \sigma dW_t.$$

Then for the function $f(x, y)$ we have

$$df(t,S) = \left(\mu f_y(t,S) + f_x + \frac{1}{2}\sigma^2 f_{yy}(t,S)\right)dt + \sigma f_y(t,S)dW_t$$

or

$$df(t,S) = \left(f_x(t,S) + \frac{1}{2}\sigma^2 f_{yy}(t,S)\right)dt + f_y(t,S)dS_t.$$

1. The special case of the Taylor expansion (informal derivation)

Expanding $\Delta f(t, S)$ gives

$$\Delta f(t,S) = f_x(t,S)\Delta t + f_y(t,S)\Delta S + \frac{1}{2}f_{xx}(t,S)\left(\Delta t\right)^2 + \frac{1}{2}f_{yy}(t,S)\left(\Delta S\right)^2 + f_{xy}(t,S)\Delta t\Delta S + \ldots$$

or in differential form

$$df(t,S) = f_x(t,S)dt + f_y(t,S)dS + \frac{1}{2}f_{xx}(t,S)\left(dt\right)^2 + \frac{1}{2}f_{yy}(t,S)\left(dS\right)^2 + f_{xy}(t,S)dtdS + \ldots$$

as $\left(dt\right)^2 \to 0$ and $\left(dS_t\right)^2 = \left(\sigma dW_t\right)^2 = \sigma^2 dt$ because of the quadratic variation of the Brownian Motion. The rest of the quadratic and higher terms converges also to zero by approximating $dW_t \approx \sqrt{dt}$. Then

$$df(t,S) = f_x(t,S)dt + f_y(t,S)dS + \frac{1}{2}\sigma^2 f_{yy}(t,S)dt$$

or

$$df(t,S) = \left(f_x(t,S) + \frac{1}{2}\sigma^2 f_{yy}(t,S)\right)dt + f_y(t,S)dS_t$$

which is the Ito formula from above.

2. General case of the chain rule

It is known, that for differentiable functions the following holds: for $g : \mathbb{R} \to \mathbb{R}^2$ and $f : Im\left(g\right) \supseteq V \to \mathbb{R}$

$$\frac{d}{dt} f\left(g\left(t\right)\right) = f_x\left(g_1\left(t\right), g_2\left(t\right)\right) \frac{dg_1}{dt}\left(t\right) + f_y\left(g_1\left(t\right), g_2\left(t\right)\right) \frac{dg_2}{dt}\left(t\right).$$

Supposing the that S_t is differentiable and setting $g\left(t\right) = \left(t, S_t\right)$, the above formula yields

$$\frac{d}{dt} f\left(t, S_t\right) = f_x\left(t, S_t\right) \frac{dt}{dt} + f_y\left(t, S_t\right) \frac{dS_t}{dt}$$

or in differential form

$$df\left(t, S_t\right) = f_x\left(t, S_t\right) dt + f_y\left(t, S_t\right) dS_t.$$

This equation shows also the basic property of a differentiable function: **it is locally approximated by a linear function.** This property is fundamental: differentiable functions have 0 quadratic variation, that is: they change their direction very slowly, that's why they can be linearly approximated. This does not hold for the Brownian Motion, which changes its direction very fast, having quadratic variation $\langle W_t \rangle = t$. Thus the approach over the Taylor expansion for a differential function S_t would make the $\frac{1}{2}\sigma^2 f_{yy}(t, S)$ term disappear and yield the linear equation of the chain rule. Or on the other hand, if S_t were differentiable, it would already have $\sigma = 0$ in the dS_t formula and the Ito formula would provide again the above linear equation.